Incomplete Information: Part I

Static Games

Games in which the payoffs are not common knowledge are known as games of incomplete information. John Harsanyi won the Nobel Prize in economics for devising a method for analyzing games of incomplete information. He proposed treating a player who has different payoffs under different circumstances as a player of different types. The game is then modeled as though 'nature' moves first and chooses that player's type. The process of adding 'nature' as a player in the game is known as the Harsanyi transformation. In this kind of game a player must form beliefs about the strategy that an opponent will play and the player must also form some belief about the type of game she is playing.

The payoffs in this game of entry are presented in the following table. The beige/yellow cells correspond to the payoffs when the cost of expansion by C-Foam are high. The greenish cells correspond to the payoffs when C-Foam's cost of expansion is low.

		C-Foam			
		Low Cost of Expansion		High Cost of Expansion	
		Expand	Don't Expand	Expand	Don't Expand
HangTen	Enter	-1, 2	1, 1	-1, -1	1,1
	Stay Out	0, 4	0, 3	0, 0	0, 3

C-Foam knows his cost of expansion, and hence his payoffs. C-Foam's payoffs also depend on whether HangTen enters the market.

HangTen's payoffs depend only on whether or not she enters. These payoffs are common knowledge and so are known to both HangTen and C-Foam.

Recall that a static game of complete information can be described by three lists. What are they?

We shall refer to the game we are developing here as a static Bayesian game. It is static since C-Foam's decision to expand and HangTen's decision to enter occur simultaneously. It is Bayesian in the sense that HangTen must formulate a belief about the likelihood that C-Foam is 'high cost' before the game is actually played and C-Foam expands or not.

A static Bayesian game requires five lists to characterize it. The lists and their content for the C-Foam -- HangTen example, are

1. The list of players: C-Foam and HangTen.

2. The list of moves for each player. A move tells the player what to do when his type has been identified. For example, C-Foam should expand if the cost of doing so is low, and don't expand if the cost of doing so is high: C-Foam - *expand, don't expand*. HangTen - *enter, stay out*. A list containing one move for each player is a move profile. In this game there are four move profiles: {*expand, enter*}, {*expand, stay out*}, {*don't expand, enter*}, and {*don't expand, stay out*}.

3. The list of possible types for each player. C-Foam - *Low Cost, High Cost*, HangTen - *normal*. There are two type profiles: [*normal, low cost*] and [*normal, high cost*].

4. The list of probabilities associated with the player type profiles. The probability of the type profile [*normal, low cost*] is 2/3 for our example.

θ_H	θ_H
$p(\theta_H \mid \theta_C)$	1

θ_C	θ_{CL}	θ_{CH}
$p(\theta_C \mid \theta_H)$	2/3	1/3

5. The list of payoffs is associated with the *move* profile list and the *type* profile list.

- Since the payoff functions, possible types, and the prior probability distribution are common knowledge, we can compute expected payoffs of player i of type θ_i as

$$U\left(s_i', s_{-i}(\cdot), \theta_i\right) = \sum_{\theta_{-i}} p(\theta_{-i} \mid \theta_i) u_i(s_i', s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

when types are finite

$$= \int u_i(s_i', s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) P(d\theta_{-i} \mid \theta_i)$$

when types are not finite.

For our surfboard example we have $U_{HangTen}$(*enter, don't expand, normal, low cost*) = **1**

Now we have to 'play' the game. A player's pure strategy consists of a *move* as a function of the player's *type*. HangTen's strategies include {enter(normal) and stay out(normal)}. The list of strategies for C-Foam are {expand(low cost), don't expand(low cost), expand(high cost), don't expand(high cost)}. If we define a strategy profile as a doublet that includes one strategy from each player then there are eight strategy profiles. Equipped with the strategy lists for each player and the knowledge that there are eight strategy profiles we can write the strategic form of the the game as

		C-Foam			
		Expand(lo), Expand(hi)	**Expand(lo), Don't expand(hi)**	**Don't expand(lo), Expand(hi)**	**Don't expand(lo), Don't expand(hi)**
HangTen	**Enter**				
	Stay Out				

There are no payoffs in the table at this point. Each strategy for HangTen consists of a single move. Each strategy for C-Foam consists of a pair of moves. This provides us with an enumeration of all the plans for C-Foam regardless of where in the game the players wind up. The idea is similar in spirit to the plans and profiles we put together in our analysis of subgame perfection. Hence, the payoffs in each cell will consist of three numbers, rather than the two we have been accustomed to seeing in strategic form games heretofore.

If HangTen plays **Enter**, and C-Foam plays **{Expand(lo), Expand(hi)}** then the payoffs are (**(-1)**, **(2, -1)**). We can see from the first table that when HangTen plays enter and C-Foam expands, regardless of its type, then HangTen earns -1. Upon HangTen's entry, C-Foam earns 2 if it is low cost and expands, but earns -1 if it is high cost and expands.

Why do we enter both payoffs for C-Foam? Remember, HangTen needs to form some conjecture about C-Foam's behavior, but doesn't know C-Foam's type. Therefore HangTen needs to consider all of C-Foam's contingencies for one of its own strategies.

To continue: If HangTen plays **Stay Out** and C-Foam plays **{Don't expand(lo), Don't expand(hi)}** then the payoffs are (**(0)**, **(3, 3)**). If

HangTen plays Stay out then its payoff is always zero. If HangTen plays Stay out then C-Foam's payoff is 3 whenever it doesn't expand. You can find these payoffs in the last row, second and fourth columns of the first table.

If HangTen plays **Enter** and C-Foam plays **{Expand(lo), Don't expand(hi)}** then the payoffs are (**(-1/3)**, **(2, 1)**). Looking at the first table it is easy to see where the pair **(2, 1)** came from. Upon HangTen's entry C-Foam will earn 2 if they expand when they are low cost, and will earn 1 if they don't expand when they are high cost. Where did the **-1/3** come from?

HangTen is playing Enter, but doesn't know the *type* of C-Foam. If HangTen plays Enter and C-Foam expands whenever they are low cost, then HangTen earns -1. If HangTen plays Enter and C-Foam does not expand whenever they are high cost, then HangTen will earn 1. HangTen's prior belief is that C-Foam is low cost with probability 2/3 and high cost with probability 1/3. Therefore HangTen's expected payoff is 2/3(-1) + 1/3(1) = -1/3.

Let's look at {**Enter**, **(Don't expand(lo), Expand(hi))**}. The payoff pair for C-Foam is, from the first table, **(1, -1)**. HangTen will earn 1 if they enter and C-Foam does not expand even though they are low cost. HangTen will earn -1 if they enter and C-Foam expands even though they are high cost. Again, HangTen doesn't know C-Foam's type, but their prior belief is that C-Foam is low cost with probability 2/3. Therefore, HangTen's expected payoff is 2/3(1) + 1/3(-1) = 1/3.

Cutting forward, we can fill in the blanks in the payoff table.

		C-Foam			
		Expand(lo), Expand(hi)	**Expand(lo), Don't expand(hi)**	**Don't expand(lo), Expand(hi)**	**Don't expand(lo), Don't expand(hi)**
HangTen	**Enter**	((-1), (2, -1))	((-1/3), (2, 1))	((1/3), (1, -1))	((1), (1, 1))
	Stay Out	((0), (4, 0))	((0), (4, 3))	((0), (3, 0))	((0), (3, 3))

From HangTen's perspective, does C-Foam have a dominant strategy? Compare the blue payoffs for C-Foam column-wise. C-Foam's **Expand(lo), Don't expand(hi)** strategy always does better than **Expand(lo), Expand(hi)**. Both entries in the pair **(2, 1)** are at least as large as the entries in the pair **(2, -1)** and the entries in the pair **(4, 3)** are at least as large as the entries in the pair **(4, 0)**. Indeed, proceeding in this fashion, HangTen believes that C-Foam does better with the strategy **Expand(lo), Don't expand(hi)** than with any other of its possible strategies. With this in mind HangTen can see that its expected payoff from **Stay out (0)** is better than its expected payoff from enter **(-1/3)**.

The solution to the game {Stay Out, (Expand(lo), Don't expand(hi))}, highlighted in yellow, is known as a Bayes Nash Equilibrium. The player's strategy is a best response to the strategies of the other players, whatever the players type.

Final Notes:

1. Notice the beauty of the Harsanyi Transformation. We have taken a game of incomplete information and turned it into a game of complete but imperfect information!

2. You can interpret a game of incomplete information as a game of mixed strategies. Can you explain this?

3. The concepts used in static games with incomplete information are used to analyze **auctions.**

Be sure to distinguish imperfect and incomplete information.
1. A list of players, 2. A list of strategies for each player, and 3. A list of payoffs for each player for every possible collection of strategies.

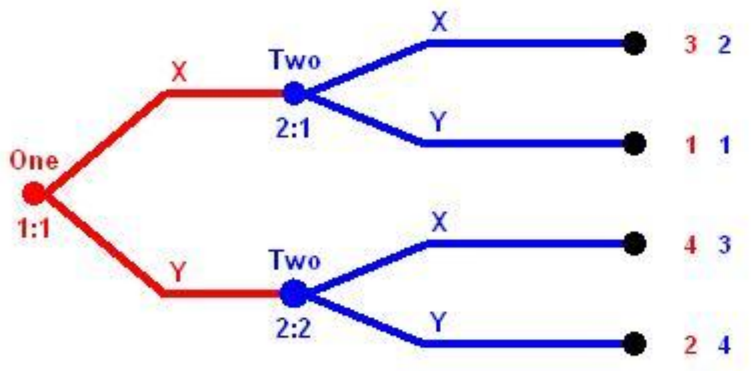
Static Games of Incomplete Information: Part II

As a starting point consider the following game of imperfect information between two players

		Player 2	
		X	Y
Player 1	X	3,2	1, 1
	Y	4,3	2, 4

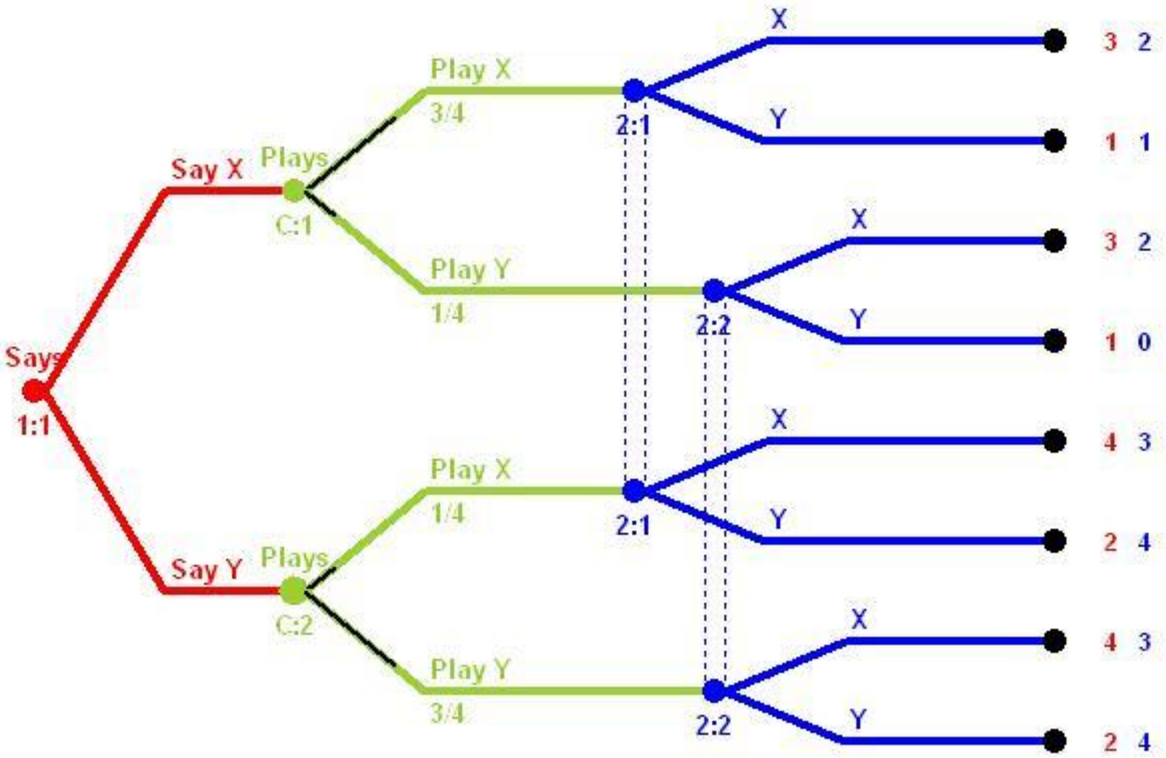
Player 1 has the dominant strategy of playing Y. Eliminating the dominated strategy leads to Player 2 playing Y. The solution to the game is <Y, Y>

Now suppose that Player 1 moves first, and Player 2 moves second knowing what 1 has played; a dynamic game of perfect information.



Using roll back induction the solution to the game is strategy X by Player 1 and then X by player 2, denoted <X, X>. Note that this was not even a Nash Equilibrium in the Normal Form Game! This happens because Player 2 cannot make a credible commitment to always play Y.

Now suppose that Player 1 is known to be a liar. Although Player 1 announces the strategy he intends to play, he might not be telling the truth. Fortunately, Player 2 knows that Player 1 tells the truth 75% of the time. The game tree now becomes.



Reading from left to right, says he will play strategy X. With 75% probability he truthfully plays the announced strategy. One has told the truth and Two plays X then the payoffs are (3, 2). Alternatively, if One says he will play X but he lies about it and Plays Y, whereupon Two plays, say, Y and the payoffs are (1, 0). Imagine Player On has two cards; one labeled X and one labeled Y. He says X, but lays the card on the table face down; he alone knows that he has put Y on the table. This is a game of imperfect information since Two doesn't know if she is playing against a truthful opponent or a deceitful opponent. As an interim step we can write the situation as follows:

		Player One				
		Truthful (p = 3/4)			Deceitful (1-p = 1/4)	
		Say X and Play X	Say Y and Play Y		Say Y and Play X	Say X and Play Y
Player	X	2, 3	3, 4		3, 4	2, 3
Two	Y	1, 1	4,2		4, 2	0, 1

We'll turn this set of side-by-side games into a Normal Form game in which Player 1 chooses from among 'moves' and Player 2 chooses a strategy.

		Player One			
		X(T), X(D)	**X(T), Y(D)**	**Y(T), X(D)**	**Y(T), Y(D)**
Player Two	**X**	2(3/4)+3(1/4)=**9/4, {3, 4}**	2(3/4)+2(1/4)=**2, {3, 3}**	3(3/4)+3(1/4)=**3, {4, 4}**	3(3/4)+2(1/4)=**11/4, {4, 3}**
	Y	1(3/4)+4(1/4)=**7/4, {1, 2}**	1(3/4)+0(1/4)=**13/4, {1, 1}**	4(3/4)+4(1/4)=**4, {2, 2}**	4(3/4)+0(1/4)=**3, {2, 1}**

Player one still goes first, but Two's information is incomplete since she doesn't know what sort of Player One really is. Look at Player 1. He can never do better than playing Y if truthful and playing X when deceitful. With that in mind, Player Two sees that she should play Y.