به نام خدا

وزارت علوم، تحقیقات و فناوری
دانشکده علوم اقتصادی

An introduction to Game Theory

Lecture 1
Introduction

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References

1. Joseph Harrington, Games, Strategies, and Decision Making, Worth Publishers (2008) (ISBN-13: 978-0-7167-6630-8)
2. Eric Rasmusen, Games and Information: An Introduction to Game Theory (2005)
3. Martin Osborne, An Introduction to Game Theory, Oxford University Press (2003).

Evaluation

Course Work	60%
Final Examination	40%

Important dates

زمان	وظایف
92/08/28	امتحان میان ترم
92/08/15	انتخاب موضوع تحقیق
	امتحان پایان ترم

Presentation Outline

- Introduction
- Static Games of Complete and perfect Information
- Competitive Games
- The Methods for Solving Games
- Dynamic Games of Complete and Perfect Information

A general introduction to game theory, its origins, and classifications

What is Game Theory?

- A formal way to analyze interaction among a group of rational agents who behave strategically
- A large number of situations that confront us in our day to day lives can be thought of as “games” with us as “players”
- And they can be analyzed using the tools of game theory

Fundamental Principles of Game Theory

1. Each player makes the best possible move.
2. Each player knows that his or her opponent is also making the best possible move.

What is Game Theory?

	One player	Many players
Static	Mathematical programming	(Static) game theory
Dynamic	Optimal control theory	Dynamic (and/or differential) game theory

What is game problems?

Game problems continuously occur in our daily lives such as: trading, marketing,

automobile insurance, advertising, football game and so on.

In fact, any problem models a situation where two or more players operate in a same environment, with different (possibly confecting) goals can be considered as a game problem.

This is natural that each player in a game wants to determine the best decision or the best strategy for achieving the aim. Thus the decision makers are considered as players.

A strategy is a function that tells how to select one of his control function whenever he must make a move for all possible events that may have occurred. Games are usually of situations of conflict or cooperation. When the objectives of the players are cooperative, the game is called a cooperative game; when they are conflicting, the game is called a conflict game or non cooperative; when some are cooperative and some are conflicting, the game is called a mixed game.

Games in everyday life

- Tennis players deciding whether to serve to the forehand or backhand of their opponent
- The local bakery offering a discounted price on pastries just before it closes
- Employees deciding how hard to work when the boss is away
- Friends choosing where to go have dinner
- Parents trying to get children to behave
- Commuters deciding how to go to work
- Businesses competing in a market
- Diplomats negotiating a treaty
- Gamblers betting in a card game

Games in everyday life

- Persian rug seller deciding how quickly to lower the price when haggling with a tourist
- Airline companies trying to decide whether to cut prices or not
- Qantas and Air New Zealand trying to decide whether to merge or not
- And the response of competitors to a merger

Examples

The best way to understand which situations can be modelled as games and which cannot is to think about examples like the following:

1. OPEC members choosing their annual output;
2. General Motors purchasing steel from USX;
3. Two manufacturers, one of nuts and one of bolts, deciding whether to use metric or American standards;
4. A board of directors setting up a stock option plan for the chief executive officer;
5. The US Air Force hiring jet fighter pilots;
6. An electric company deciding whether to order a new power plant given its estimate of demand for electricity in ten years.

Who is this?

John Nash

One of the early researchers
in game theory,
his work resulted in a form of
equilibrium named after him

Historical Development

In 1928, John von Neumann himself had resolved completely an open fundamental problem in zero-sum games, that every finite two-player zero-sum game admits a saddle point in mixed strategies, which is known as the ***Minimax Theorem*** [1] a result which Emile Borel [2] had conjectured to be false eight years before.

The Minimax Theorem: For every two-person, zero-sum game, there exists an equilibrium strategy.

[1] J. von Neumann, Zur theorie der Gesellschaftspiele," Mathematische Annalen, 100:295-320, 1928.

[2] E. Borel (1913) mixed strategies, conjecture of non-existence

Historical Development

The classic book [3] by John von Neumann and Morgenstern is widely regarded as the starting point of the mathematical theory of games. While this book focuses on two-players, zero-sum games, the later paper of Nash [4] provided a concept of solution for general non-cooperative games for N players. The monograph by Stackelberg [5] provided a further contribution to the theory of games, motivated by the analys is of market economy.

[3] J. von Neumann and O. Morgenstern, Theory of Games and Economic Third edition. Princeton University Press, 1980 (First edition 1944). Behavior.

[4] J. Nash, Non-cooperative games, Annals of Math. 2 (1951), 286-295.

[5] H. von Stackelberg, The Theory of the Market Economy. Oxford Univ. Press, 1952.

Historical Development...

- E. Zermelo (1913) chess, the game has a solution, solution concept: backwards induction.
- E. Borel (1913) mixed strategies, conjecture of non-existence.
- J. v. Neumann (1928) existence of solutions in zero-sum games.
- J. v. Neumann and O. Morgenstern (1944) Theory of Games and Economic Behavior: Axiomatic expected utility theory, Zero-sum games, cooperative game theory.
- J. Nash (1950) Nonzero sum games and the concept of Nash equilibrium.
- R. Selten (1965,75) dynamic games, sub game perfect equilibrium.
- J. Harsanyi (1967, 68) games of incomplete information, Bayesian equilibrium.

Historical Development...

As a recognition of the vitality of the field a total of 10 **Nobel Prizes** were given in Economic Sciences for work primarily in game theory:

- John Harsanyi, John Nash, and Reinhard Selten for their pioneering analysis of equilibria in the theory of non-cooperative games. ” 1994.
- Robert Aumann and Thomas Schelling, for having enhanced our understanding of conflict and cooperation through game-theory analysis.” 2005.
- Leonid Hurwicz, Eric Maskin, and Roger Myerson, “for having laid the foundations of mechanism design theory”. 2007.
- Alvin E. Roth and Lloyd S. Shapley for the theory of stable allocations and the practice of market design.” 2012.

International Societies vs Journals

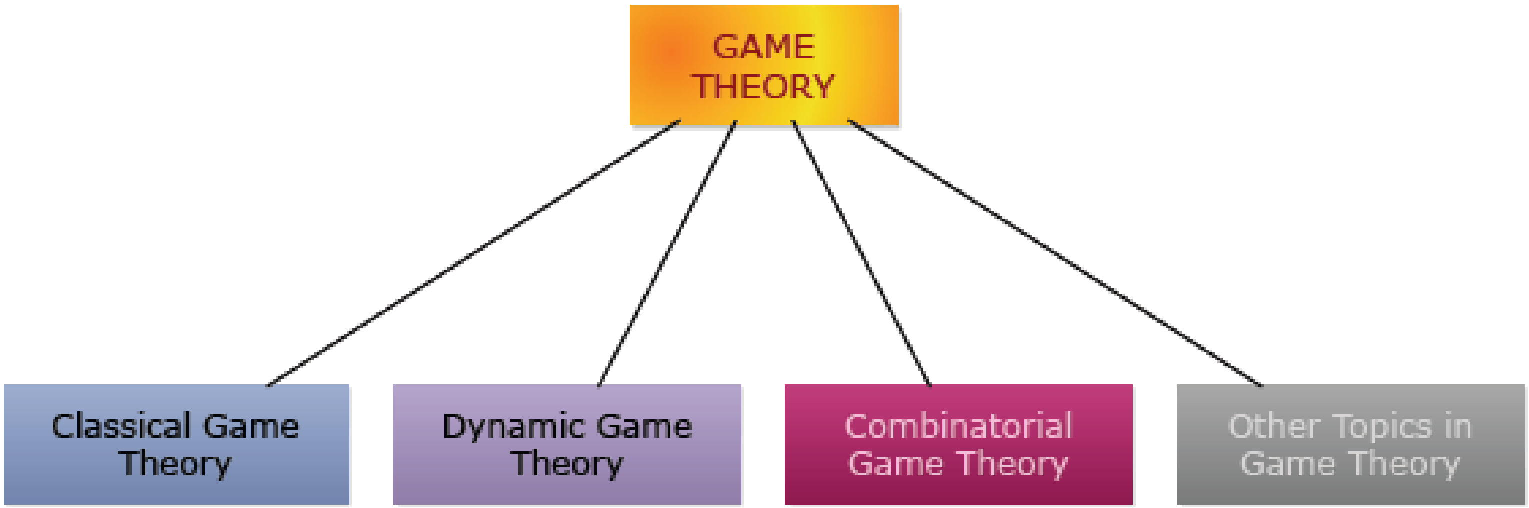
Two international societies exist:

- ❑ International Society of Dynamic Games (1990 -)
- ❑ Game Theory Society (1999 -)

There are at least four journals that are primarily on game theory:

- ❑ Games and Economic Behavior
- ❑ International J. Game Theory
- ❑ International Game Theory Review
- ❑ J. Dynamic Games and Applications

Classification of Games



Games with finite numbers of strategies.

Games with probability (either induced by the player or the game).

Games with coalitions.

Examples:
Poker, Strategic Military Decision Making, Negotiations.

Games with time.

Games with motion or a dynamic component.

Examples:
Optimal play in a dog fight. Chasing your brother across a room.

Games with no chance.

Generally two player strategic games played on boards.

Moves change the structure of a game board.

Examples:
Chess, Checkers, Go, Nim.

Evolutionary Game Theory

Experimental / Behavioral Game Theory

Examples:
Evolutionary dynamics in closed populations, Determining why altruism is present in human society.

Classification of Games

The basic distinctions can be made at the outset:

1. non-cooperative vs. cooperative games
2. strategic (or normal form) games vs. extensive (form) games
3. games with complete vs. incomplete information
4. Non-Zero Sum vs. Zero Sum Games
5. One-Shot vs. Repeated Games

Four elements in every game

- Players

  Players are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.

- Strategies available to each player (Actions)
- Payoffs

  Based on your decision(s) and the decision(s) of other(s)

- Information

$$G = G(P_i, S_i, U_i), \quad i = 2, 3, \ldots, N$$

Glossary

❑ ***Backward induction***

Backward induction is a technique to solve a game of perfect information. It first considers the moves that are the last in the game, and determines the best move for the player in each case. Then, taking these as given future actions, it proceeds backwards in time, again determining the best move for the respective player, until the beginning of the game is reached.

❑ ***Common knowledge***

A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

❑ ***Dominating strategy***

A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

❑ ***Extensive game***

An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

Glossary

❑ ***Game***

A game is a formal description of a strategic situation.

❑ ***Game theory***

Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

❑ ***Mixed strategy***

A mixed strategy is an active randomization, with given probabilities, that determines the player's decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

❑ ***Nash equilibrium***

A Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change his strategy and get a better payoff.

Glossary

❑ ***Player***

A player is an agent who makes decisions in a game.

❑ ***Rationality***

A player is said to be rational if he seeks to play in a manner which maximizes his own payoff. It is often assumed that the rationality of all players is common knowledge.

❑ ***Payoff***

A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

❑ ***Perfect information***

A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.

Glossary

❑ ***Strategic form***

A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

❑ ***Strategy***

In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player. The two types of strategy are

1. Pure strategy
2. Mixed strategy

❑ ***Zero-sum game***

A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.

Strategic (Normal) Form Games

Static Games of Complete and perfect Information

Static Games

- Games where players choose actions simultaneously are simultaneous move games.
  - Examples: Prisoners' Dilemma, Sealed-Bid Auctions.
  - Must anticipate what your opponent will do right now, recognizing that your opponent is doing the same.

Normal (Strategic) Form Games

- A game consists of a set of players $\{1, 2, 3, \ldots, n\}$, each with a set of strategies $S_1, S_2, \ldots, S_n$.

- The strategy space S of the game is the set of vectors or strategy profiles $S_1 \times S_2 \times \ldots \times S_n$.

- For any profile of strategies s ∈ S and any player i, there is a payoff $U_i(s)$.

Assumptions in Static Normal Form Games

- All players are rational.
- Rationality is common knowledge.
- Players move simultaneously. (They do not know what the other player has chosen).
- Players have complete but imperfect information.

Bi-Matrix Games

- Two players, Row and Column
- Row has m strategies
- Column has n strategies
- Payoffs represented an matrix A is $m \times n.$

whose entries are pairs of numbers (x, y).

- Entry $a_{ij} = (x, y)$ means that when Row plays i

and Column plays j, the payoff to Row is x and the payoff to Column is y.

Prisoners' Dilemma

$$S_1 = \{a_1, a_2\},\ S_2 = \{a_1, a_2\}$$

	Prisoner 2: a1	Prisoner 2: a2
Prisoner 1: a1	β, β	θ, α
Prisoner 1: a2	α, θ	γ, γ

$$\theta < \gamma < \beta < \alpha$$

Prisoners' Dilemma

- Two players, prisoners 1, 2.
- Each prisoner has two possible actions.
  - Prisoner 1: Don't Confess, Confess
  - Prisoner 2: Don't Confess, Confess
- Players choose actions simultaneously without knowing the action chosen by the other.
- Payoff consequences quantified in prison years.
  - If neither confesses, each gets 1 year
  - If both confess, each gets 5 years
  - If 1 confesses, he goes free and other gets 15 years
- Fewer years=greater satisfaction=>higher payoff.
  - Prisoner 1 payoff first, followed by prisoner 2 payoff.

The Prisoners' Dilemma Game

Payoff = (years in prison)

Each player has only two strategies, each of which is a single action

- ❑ Non-zero-sum
- ❑ Imperfect information: neither player knows the other’s move until after both players have moved
- ❑ Simultaneously Game

Prisoners' Dilemma

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	5, 5	15, 0
Don't Confess	0, 15	1, 1

Battle of Sexes

- A couple deciding how to spend the evening
- Wife would like to go for a Opera
- Husband would like to go for a Football
- Both however want to spend the time together
- Scope for strategic interaction

A Coordination Game
Battle of the Sexes

Husband

Wife

	Opera	**Football**
Opera	2, 1	0, 0
Football	0, 0	1, 2

A Strictly Competitive Game
Matching Pennies

Player 2

Player 1

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

No NE in pure strategies

Strategies

Suppose the agents agent 1, agent 2, …, agent n

- For each i, let S_i = {all possible strategies for agent i}
  s_i will always refer to a strategy in S_i
- A strategy profile is an n-tuple $S = (s_1, s_2, \ldots, s_n)$, one strategy for each agent
- Utility $U_i(s) = \text{payoff}$ for agent i if the strategy profile is S
- s_i strongly dominates s_i' if agent i always does better with s_i than s_i'

$$\forall\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n,$$
$$U_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) > U_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

- s_i weakly dominates s_i' if agent i never does worse with s_i than s_i' and there is at least one case where agent i does better with s_i than s_i',

$$\forall\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n,\ U_i(\ldots, s_i, \ldots) \geq U_i(\ldots, s_i', \ldots)$$
and
$$\exists\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n\ \ U_i(\ldots, s_i, \ldots) > U_i(\ldots, s_i', \ldots)$$

Dominant Strategy Equilibrium

- s_i is a (**strongly, weakly**) **dominant** strategy if it (strongly, weakly) dominates every $s_i' \in S_i$
- **Dominant strategy equilibrium**:
  - A set of strategies $(s_1, \ldots, s_n)$ such that each s_i is dominant for agent i
  - Thus agent i will do best by using s_i rather than a different strategy, regardless of what strategies the other players use
  - In the prisoner's dilemma, there is one dominant strategy equilibrium: both players defect

Agent 2

Agent 1

	Action C	Action D
Action C	5, 5	15, 0
Action D	0, 15	1, 1

Pure and Mixed Strategies

- **Pure strategy**: select a single action and play it
  - Each row or column of a payoff matrix represents both an action and a pure strategy
- **Mixed strategy**: randomize over the set of available actions according to some probability distribution
  - Let $A_i = \{\text{all possible actions for agent } i\}$, and a_i be any action in A_i
  - $s_i(a_j)$ = probability that action a_j will be played under mixed strategy s_i
- The **support** of s_i is
  - support(s_i) = {actions in A_i that have probability > 0 under s_i}
- A pure strategy is a special case of a mixed strategy
  - support consists of a single action
- **Fully mixed strategy**: every action has probability > 0
  - i.e., support(s_i) = A_i

Mixed Strategies

A game in strategic form does not always have a Nash equilibrium in which each player deterministically chooses one of his strategies. However, players may instead randomly select from among these pure strategies with certain probabilities. Randomizing one's own choice in this way is called a mixed strategy. Nash showed in 1951 that any finite strategic-form game has an equilibrium if mixed strategies are allowed. As before, an equilibrium is defined by a (possibly mixed) strategy for each player where no player can gain on average by unilateral deviation. Average (that is, expected) payoffs must be considered because the outcome of the game may be random.

Expected Utility

- A payoff matrix only gives payoffs for pure-strategy profiles
- Generalization to mixed strategies uses ***expected utility***
- Let $S = (s_1, \ldots, s_n)$ be a profile of mixed strategies
  - For every action profile $(a_1, a_2, \ldots, a_n)$, multiply its probability and its utility
    - $U_i(a_1, \ldots, a_n)\, s_1(a_1)\, s_2(a_2) \ldots s_n(a_n)$
  - The expected utility for agent i is

$$U_i(s_1,\ldots,s_n) = \sum_{(a_1,\ldots,a_n)\in \mathbf{A}} U_i(a_1,\ldots,a_n)\; s_1(a_1)\; s_2(a_2)\; \ldots\; s_n(a_n)$$

Best Response

- Some notation:
  - If $S = (s_1, \ldots, s_n)$ is a strategy profile, then $S_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,
    - i.e., S_{-i} is strategy profile S without agent i's strategy
  - If s_i' is any strategy for agent i, then
    - $(s_i', S_{-i}) = (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$
  - Hence $(s_i, S_{-i}) = S$
- s_i is a **best response** to S_{-i} if

  $U_i(s_i, S_{-i}) \geq U_i(s_i', S_{-i})$ for every strategy s_i' available to agent i
- s_i is a **unique** best response to S_{-i} if

  $U_i(s_i, S_{-i}) > U_i(s_i', S_{-i})$ for every $s_i' \neq s_i$

Nash Equilibrium

- A strategy profile $s = (s_1, \ldots, s_n)$ is a **Nash equilibrium** if for every i,
  - s_i is a best response to S_{-i} , i.e., no agent can do better by unilaterally changing his/her strategy

- **Theorem (Nash, 1951)**: Every game with a finite number of agents and action profiles has at least one Nash equilibrium

- In the Prisoner's Dilemma, (D,D) is a Nash equilibrium
  - If either agent unilaterally switches to a different strategy, his/her expected utility goes below 1

- A dominant strategy equilib ium is always a Nash equilibrium

	Action C	Action D
Action C	5, 5	15, 0
Action D	0, 15	1, 1

Game theory: Payoff matrix

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- A payoff matrix shows the payout to each player, given the decision of each player

How do we interpret this box?

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- The first number in each box determines the payout for Person 1
- The second number determines the payout for Person 2

How do we interpret this box?

		Person 2	
		Action C	Action D
Person 1	Action A	10, 2	8, 3
	Action B	12, 4	10, 1

- Example
  - If Person 1 chooses Action A and Person 2 chooses Action D, then Person 1 receives a payout of 8 and Person 2 receives a payout of 3

How do we find Nash equilibrium (NE)?

- Step 1: Pretend you are one of the players
- Step 2: Assume that your "opponent" picks a particular action
- Step 3: Determine your best strategy (strategies), given your opponent's action
  - Underline any best choice in the payoff matrix
- Step 4: Repeat Steps 2 & 3 for any other opponent strategies
- Step 5: Repeat Steps 1 through 4 for the other player
- Step 6: Any entry with all numbers underlined is NE

Steps 1 and 2

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- Assume that you are Person 1
- Given that Person 2 chooses Action C, what is Person 1's best choice?

Step 3

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- Underline best payout, given the choice of the other player
- Choose Action B, since 12 > 10 ➔ underline 12

Step 4

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

Person 2 (columns); Person 1 (rows)

- Now assume that Person 2 chooses Action D
- Here, 10 > 8 ➔
- Choose and underline 10

Step 5

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- Now, assume you are Person 2
- If Person 1 chooses A
  - 3 > 2 ➔ underline 3
- If Person 1 chooses B
  - 4 > 1 ➔ underline 4

Step 6

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- Which box(es) have underlines under both numbers?
  - Person 1 chooses B and Person 2 chooses C
  - This is the only NE

Double check our NE

Person 2

Person 1	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- What if Person 1 deviates from NE?
  - Could choose A and get 10
  - Person 1's payout is lower by deviating 👍

Double check our NE

Person 2

Person 1

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

- What if Person 2 deviates from NE?
  - Could choose D and get 1
  - Person 2's payout is lower by deviating 👍

Dominant strategy

Person 2

	Action C	Action D
Action A	10, 2	8, 3
Action B	12, 4	10, 1

Person 1

- A strategy is dominant if that choice is definitely made no matter what the other person chooses
  - Example: Person 1 has a dominant strategy of choosing B

New example

Person 2

	Yes	No
Yes	20, 20	5, 10
No	10, 5	10, 10

Person 1

- Suppose in this example that two people are simultaneously going to decide on this game

New example

	Person 2	
Person 1	Yes	No
Yes	20, 20	5, 10
No	10, 5	10, 10

- We will go through the same steps to determine NE

Two NE possible

		Person 2	
		Yes	No
Person 1	Yes	20, 20	5, 10
	No	10, 5	10, 10

- (Yes, Yes) and (No, No) are both NE
- Although (Yes, Yes) is the more efficient outcome, we have no way to predict which outcome will actually occur

Two NE possible

- Additional information or actions may help to determine outcome
  - If people could act sequentially instead of simultaneously, we could see that 20, 20 would occur in equilibrium

- When there are multiple NE that are possible, economic theory tells us little about which outcome occurs with certainty.

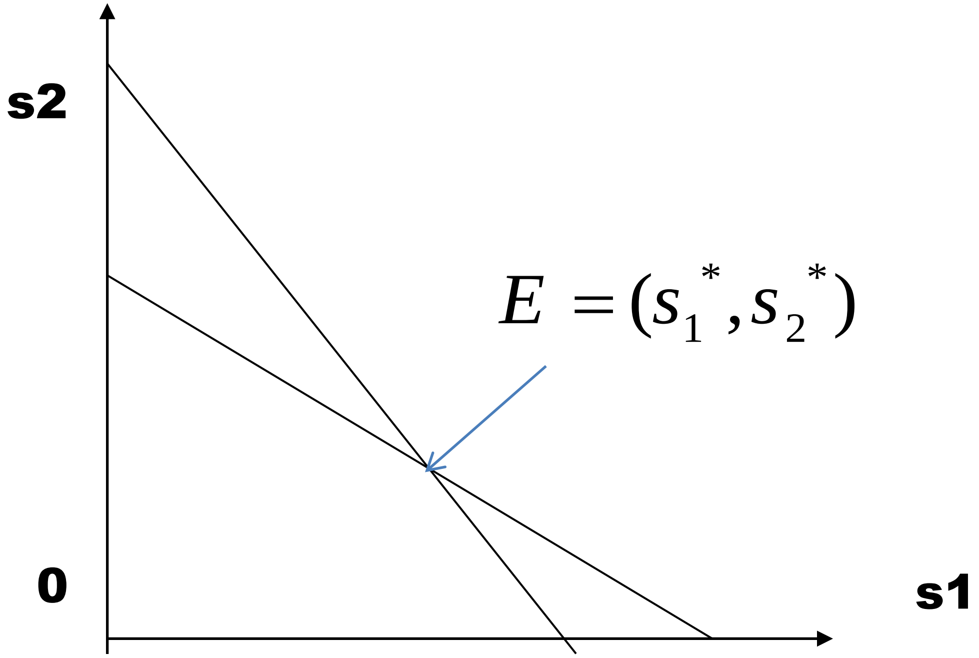
Example

$$N = \{1, 2\}, \quad \text{Players: } P_1, P_2$$

$$u_1 = 10s_1 - s_1^2 - s_1 s_2 - 3s_1, \quad u_2 = 10s_2 - s_2^2 - s_1 s_2 - 2s_2$$

$$s_1 = f_1(s_2) = \frac{1}{2}(7 - s_2), \quad s_2 = f_2(s_1) = \frac{1}{2}(8 - s_1)$$

$$s_1^* = 2, \quad s_2^* = 3$$



Finding Mixed Strategies Equilibria

- Generally it's tricky to compute mixed-strategy equilibria
  - But easy if we can identify the support of the equilibrium strategies

- Suppose a best response to S_{-i} is a mixed strategy s whose support includes ≥ 2 actions
  - Then every action a in support(s) must have the same expected utility $U_i(a,S_{-i})$
    - If some action a^* in support(s) had a higher expected utility than the others, then it would be a better response than s
  - Thus *any* mixture of the actions in support(s) is a best response

Example: Battle of Sexes

- **Battle of the Sexes**
  - Two agents need to coordinate their actions, but they have different preferences
  - Original scenario:
    - husband prefers football
    - wife prefers opera
  - Another scenario:
    - Two nations must act together to deal with an international crisis
    - They prefer different solutions

Husband

Wife

	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

Nash equilibria

- This game has two pure-strategy Nash equilibria (circled above) and one mixed-strategy Nash equilibrium
  - How to find the mixed-strategy Nash equilibrium?

Example: Battle of Sexes

- Suppose both agents randomize, and the husband's mixed strategy s_h is

  $s_h(\text{Opera}) = p; \quad s_h(\text{Football}) = 1 - p$

- Expected utilities of the wife's actions:

  $U_w(\text{Football}, s_h) = 0p + 1(1 - p)$

  $U_w(\text{Opera}, s_h) = 2p$

husband

wife

	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

- If the wife mixes between her two actions, they must have the same expected utility
  - If one of the actions had a better expected utility, she'd do better with a pure strategy that *always* used that action
  - Thus $0p + 1(1 - p) = 2p$, so $p = 1/3$
- So the husband's mixed strategy is $s_h(\text{Opera}) = 1/3; \quad s_h(\text{Football}) = 2/3$

Example: Matching Pennies

Matching Pennies

- Each agent has a penny
- Each agent independently chooses to display his/her penny heads up or tails up
- Easy to see that in this game, no pure strategy could be part of a Nash equilibrium
  - For each combination of pure strategies, one of the agents can do better by changing his/her strategy
    - for (Heads,Heads), agent 2 can do better by switching to Tails
    - for (Heads,Tails), agent 1 can do better by switching to Tails
    - for (Tails,Tails), agent 2 can do better by switching to Heads
    - for (Tails,Heads), agent 1 can do better by switching to Heads
- But there's a mixed-strategy equilibrium:
  - (s,s), where $s(\text{Heads}) = s(\text{Tails}) = ½$

P2

P1

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1, -1

No NE in pure strategies

Example

$$S_1 = \{p :\ 0 \le p_i \le 1, p_1 + p_2 = 1\}$$

$$S_2 = \{q :\ 0 \le q_i \le 1,\ q_1 + q_2 = 1\}$$

$$E(u_1) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j u_{1ij}$$

$$E(u_2) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j u_{2ij}$$

P2

P1

	C1	C2
R1	1,4	4,1
R2	3,2	2, 3

$$E(u_1) = p_1 q_1 + 4 p_1 q_2 + 3 p_2 q_1 + 2 p_2 q_2$$

$$E(u_2) = 4 p_1 q_1 + p_1 q_2 + 2 p_2 q_1 + 3 p_2 q_2$$

$$p^* = 0.25,\quad q^* = 0.5$$

$$E(u_1) = 2 + 2p + q - 4pq$$

$$E(u_2) = 3 - 2p - q + 4pq$$

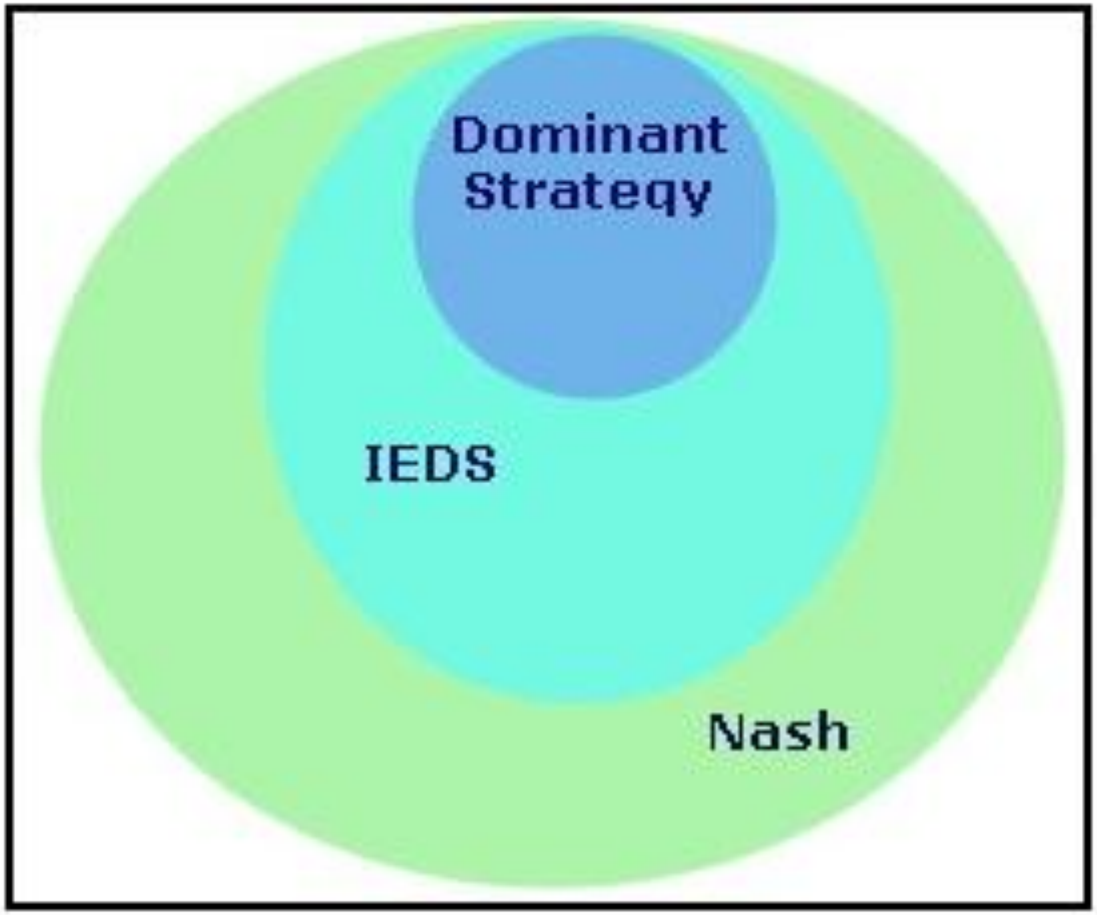
How to reason about games?

- ✓ In single-agent decision theory, look at an optimal strategy
- ✓ Maximize the agent’s expected payoff in its environment

- ✓ With multiple agents, the best strategy depends on others’ choices
- ✓ Deal with this by identifying certain subsets of outcomes called solution concepts

Some solution concepts:

- ❑ Dominant strategy equilibrium
- ❑ Iterated elimination of strictly dominated strategies
- ❑ Nash equilibrium


Dominant
Strategy
IEDS
Nash

Solution Methods of a Static Normal Form Game

- ***Equilibrium in Strictly Dominant Strategies***
- ***Iterated Elimination of Strictly Dominated Strategies (IESDS Method)***
- ***Nash Equilibrium (NE)***

Solution of a Static Normal Form Game

- ***Equilibrium in strictly dominant strategies***
  - A strictly dominant strategy is the one that yields the highest payoff compared to the payoffs associated with all other strategies.
  - Rational players will always play their strictly dominant strategies.

Solution of a Static Normal Form Game

❑ *Iterated elimination of strictly dominated strategies*

- A strategy s_i is (**strongly, weakly) dominated** for an agent i if some other strategy s'_i strictly dominates s_i
- A strictly dominated strategy can't be a best response to any move
  - So we can eliminate it (remove it from the payoff matrix)
- Once a pure strategy is eliminated, another strategy may become dominated
  - This elimination can be repeated

Solution of a Static Normal Form Game

❑ *Nash Equilibrium (NE)*

- In equilibrium neither player has an incentive to deviate from his/her strategy, given the equilibrium strategies of rival players.
- Neither player can unilaterally change his/her strategy and increase his/her payoff, given the strategies of other players.

Solution of Prisoners' Dilemma: Dominant Strategy Equilibrium

Prisoner 2

Prisoner 1

	Confess	**Don't Confess**
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

Solution of Prisoners' Dilemma: Cell-by-cell Inspection

Prisoner 2

Prisoner 1

	Confess	Don't Confess
Confess	-5, -5	0, -10
Don't Confess	-10, 0	-2, -2

NE of Prisoners' Dilemma

- The strategy profile {confess, confess} is the unique pure strategy NE of the game.
- In equilibrium both players get a payoff of –5.
- Inefficient equilibrium; (don't confess, don't confess) yields higher payoffs for both.

A Pricing Example

Firm 2

Firm 1

	High Price	Low Price
High Price	100, 100	10, 140
Low Price	140, 10	0, 0