به نام خدا

وزارت علوم، تحقیقات و فناوری
دانشکده علوم اقتصادی

An introduction to Game Theory

Lecture 2

Competitive Games

By: M. Khakestari

University of Economics Sciences

Fall 2013

Competitive Games

Non-Zero Sum versus Zero Sum Games

- A zero-sum game is one in which the players' interests are in direct conflict, e.g. in football, one team wins and the other loses; payoffs sum to zero.

  Paper, Scissors, Rock is an example of a two-person zero sum game. It is called a zero sum game because each player's loss is equal to the other player's gain.

- A game is non-zero-sum, if players interests are not always in direct conflict, so that there are opportunities for both to gain.

  An example of a non-zero sum game would be one in which the government taxed the earnings of the winner. In that case the winner's gain would be less than the loser's loss.

Two-Person and Zero-Sum Game

Two-person zero-sum games may be deterministic or probabilistic. The deterministic games will have saddle points and pure strategies exist in such games. In contrast, the probabilistic games will have no saddle points and mixed strategies are taken with the help of probabilities.

Definition of saddle point

A saddle point of a matrix is the position of such an element in the payoff matrix, which is minimum in its row and the maximum in its column.

Example Paper, Scissors, Rock

$$
\mathbf{A} \quad
\begin{array}{c}
 \\ p \\ s \\ r
\end{array}
\overset{\mathbf{B}}{
\begin{array}{ccc}
p & s & r \\
\end{array}}
$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} p & s & r \end{array} \\
\begin{array}{c} p \\ s \\ r \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Nash equilibrium : Competitive Games

Let G be a strictly competitive (zero-sum) game with N.E. In this case, the Nash equilibrium becomes the saddle-point equilibrium (SPE), which is formally defined as follows, where we leave $S = S_1 \times S_2$

A pair of actions $(s_1^*, s_2^*) \in S$ is in saddle-point equilibrium (SPE) for a game with payoff function U, if

$$U(s_1^*, s_2) \le U(s_1^*, s_2^*) \le U(s_1, s_2^*), \quad \forall (s_1, s_2) \in S$$

This also implies that the order in which minimization and maximization are carried out is inconsequential, that is

$$V^- = \max_{s_1} \min_{s_2} U(s_1, s_2) \le \min_{s_2} \max_{s_1} U(s_1, s_2) = V^+ = U(s_1^*, s_2^*) = U^*$$

where the first expression on the left is known as the upper value of the game, the second expression is the lower value of the game, and U* is known as the value of the game.

The Methods for Solving Games

All games are classified into

- Pure strategy games
- Mixed strategy games

The method for solving these two types varies. By solving a game, we need to find best strategies for both the players and also to find the value of the game.

Pure strategy games can be solved by saddle point method.

The different methods for solving a mixed strategy game are

- Dominance rule
- Simplex method

Two-Person Zero Sum Games; Reduction by Dominance

Reduction by Dominance

1. Check whether there is any row in the (remaining) matrix that is dominated by another row (this means that it is $\leq$ some other row). If there is one, delete it.
2. Check whether there is any column in the (remaining) matrix that is dominated by another column (this means that it is $\geq$ some other column). If there is one, delete it.
3. Repeat steps 1 and 2 in any order until there are no dominated rows or columns

Example

$$
\begin{array}{cc}
 & \mathbf{B} \\
\mathbf{A} &
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
1 & 3 & -1 & -1 \\
2 & 0 & 1 & 1 \\
3 & 0 & -2 & 1 \\
4 & 2 & 3 & 4
\end{array}
\end{array},
$$

$$
\begin{array}{cc}
 & \mathbf{B} \\
\mathbf{A} &
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
1 & 3 & -1 & -1 \\
4 & 2 & 3 & 4
\end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \mathbf{B} \\
\mathbf{A} &
\begin{array}{c|cc}
 & 1 & 2 \\
\hline
1 & 3 & -1 \\
4 & 2 & 3
\end{array}
\end{array}
$$

Solving Two-Person and Zero-Sum Game

Procedure to find the saddle point : **Max-Min Method**

- Select the minimum element of each row of the payoff matrix and mark them with circles.
- Select the maximum element of each column of the payoff matrix and mark them with squares.
- If their appears an element in the payoff matrix with a circle and a square together then that position is called saddle point and the element is the value of the game.

Solution of games with saddle point

To obtain a solution of a game with a saddle point, it is feasible to find out

- Best strategy for player A
- Best strategy for player B
- The value of the game

The best strategies for player A and B will be those which correspond to the row and column respectively through the saddle point.

Example: Max-Min Method

Examples

Solve the payoff matrix

1.

		Player B				
		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Solving Two-Person and Zero-Sum Game

Solution

Player B

Player A		I	II	III	IV	V	
	I	-2	0	0	5	3	-2
	II	3	2	1	2	2	1 Maximin value
	III	-4	-3	0	-2	6	-4
	IV	5	3	-4	2	-6	-6
		5	3	1	5	6	

Minimax value

Strategy of player A – II
Strategy of player B - III
Value of the game = 1

Example: Max-Min Method

Table 1. Payoff Matrix for Saddle Point Example

		Player II		
		1	2	3
	1	5	1	0
Player I	2	0	2	7
	3	4	3	4

Payoff Matrix with Saddle Point Strategies

		Player II			Minimum	Strategy
		1	2	3	of row	for II
	1	5	1	0	0	3
Player I	2	0	2	7	0	1
	3	4	3	4	3	2
Maximum for column		5	3	7		
Strategy for I		1	3	2		

Simplex Method

Let us consider the 3 x 3 matrix

$$\begin{array}{c c} & \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

$$p_1 + p_2 + p_3 = 1 \qquad q_1 + q_2 + q_3 = 1$$

Simplex Method

Step 1

Find the minimax and maximin value from the given matrix

Step 2

The objective of A is to maximize the value, which is equivalent to minimizing the value 1/V

The LPP is written as

$$\text{Min } 1/V = p_1/V + p_2/V + p_3/V$$

and constraints ≥ 1

It is written as

$$\text{Min } 1/V = x_1 + x_2 + x_3$$

and constraints ≥ 1

Similarly for B, we get the LPP as the dual of the above LPP

$$\text{Max } 1/V = Y_1 + Y_2 + Y_3$$

and constraints ≤ 1

Where $Y_1 = q_1/V$, $Y_2 = q_2/V$, $Y_3 = q_3/V$

Step 3

Solve the LPP by using simplex table and obtain the best strategy for the players

Simplex Method: Example

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

$$\begin{array}{ccl} & \text{B} & \\ \text{A} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} & \begin{array}{l} -2 \\ -1 \\ 2 \ \text{Maximin} \end{array} \\ & \begin{array}{ccc} 3 & 4 & 6 \end{array} & \\ & \text{Minimax} & \end{array}$$

LPP

Max $1/V = Y_1 + Y_2 + Y_3$

Subject to

$3Y_1 - 2Y_2 + 4Y_3 \le 1$

$-1Y_1 + 4Y_2 + 2Y_3 \le 1$

$2Y_1 + 2Y_2 + 6Y_3 \le 1$

$Y_1, Y_2, Y_3 \ge 0$

SLPP

Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3Y_1 - 2Y_2 + 4Y_3 + s_1 = 1$

$-1Y_1 + 4Y_2 + 2Y_3 + s_2 = 1$

$2Y_1 + 2Y_2 + 6Y_3 + s_3 = 1$

$Y_1, Y_2, Y_3, s_1, s_2, s_3 \ge 0$

Simplex Method: Example (cont.)

		$C_j \rightarrow$	1	1	1	0	0	0	
Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
S_1	0	1	$\boxed{3}$	-2	4	1	0	0	$1/3 \rightarrow$
S_2	0	1	-1	4	2	0	1	0	-
S_3	0	1	2	2	6	0	0	1	1/2
	1/V = 0		$\uparrow$ -1	-1	-1	0	0	0	
Y_1	1	1/3	1	-2/3	4/3	1/3	0	0	-
S_2	0	4/3	0	10/3	10/3	1/3	1	0	2/5
S_3	0	1/3	0	$\boxed{10/3}$	10/3	-2/3	0	1	$1/10 \rightarrow$
	1/V =1/3		0	$\uparrow$ -5/3	1/3	1/3	0	0	
Y_1	1	2/5	1	0	2	1/5	0	1/5	
S_2	0	1	0	0	0	1	1	-1	
Y_2	1	1/10	0	1	1	-1/5	0	3/10	
	1/V = 1/2		0	0	2	0	0	1/2	

Simplex Method: Example (cont.)

$1/V = 1/2$

$V = 2$

$y_1 = 2/5 * 2 = 4/5$

$y_2 = 1/10 * 2 = 1/5$

$y_3 = 0 * 2 = 0$

$x_1 = 0*2 = 0$

$x_2 = 0*2 = 0$

$x_3 = 1/2*2 = 1$

$S_A = (0, 0, 1)$

$S_B = (4/5, 1/5, 0)$

Value = 2

Simplex Method: Example

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} & \text{B} & \\ \text{A} & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} & \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \\ & \begin{matrix} 1 & 2 & 3 \end{matrix} & \end{array}$$

LPP

Max $1/V = Y_1 + Y_2 + Y_3$

Subject to

$2Y_1 + 0Y_2 + 0Y_3 \leq 1$

$0Y_1 + 0Y_2 + 4Y_3 \leq 1$

$0Y_1 + 3Y_2 + 0Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

SLPP

Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$2Y_1 + 0Y_2 + 0Y_3 + s_1 = 1$

$0Y_1 + 0Y_2 + 4Y_3 + s_2 = 1$

$0Y_1 + 3Y_2 + 0Y_3 + s_3 = 1$

$Y_1, Y_2, Y_3, s_1, s_2, s_3 \geq 0$

Simplex Method: Example (cont.)

		$C_j \rightarrow$	1	1	1	0	0	0	
Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
S_1	0	1	$\boxed{3}$	-2	4	1	0	0	$1/3 \rightarrow$
S_2	0	1	-1	4	2	0	1	0	-
S_3	0	1	2	2	6	0	0	1	1/2
	1/V = 0		$\uparrow$ -1	-1	-1	0	0	0	
Y_1	1	1/3	1	-2/3	4/3	1/3	0	0	-
S_2	0	4/3	0	10/3	10/3	1/3	1	0	2/5
S_3	0	1/3	0	$\boxed{10/3}$	10/3	-2/3	0	1	$1/10 \rightarrow$
	1/V =1/3		0	$\uparrow$ -5/3	1/3	1/3	0	0	
Y_1	1	2/5	1	0	2	1/5	0	1/5	
S_2	0	1	0	0	0	1	1	-1	
Y_2	1	1/10	0	1	1	-1/5	0	3/10	
	1/V = 1/2		0	0	2	0	0	1/2	

Simplex Method: Example (cont.)

$1/V = 13/12$
$V = 12/13$

$y_1 = 1/2 * 12/13 = 6/13$
$y_2 = 1/3 * 12/13 = 4/13$
$y_3 = 1/4 * 12/13 = 3/13$

$x_1 = 1/2 * 12/13 = 6/13$
$x_2 = 1/4 * 12/13 = 3/13$
$x_3 = 1/3 * 12/13 = 4/13$

$S_A = (6/13, 3/13, 4/13)$
$S_B = (6/13, 4/13, 3/13)$

$\text{Value} = 12/13 - C = 12/13 - 1 = -1/13$

Summary

- Although Nash equilibria do not always exist, one can give a guarantee, when we randomize finite games: For every finite strategic game, there exists a Nash equilibrium in mixed strategies
- A mixed strategy is a profile of probabilities over the pure strategies
- For instance, in the battle of the sexes, it is a probability to choose Opera.
- To compute the payoff a mixed strategy, you sum the payoff times the probabilities of those payoffs
- Dominant strategies can only be in pure strategies
- A pure strategy can be dominated by a mixed strategy. But you only care about whether it is dominated against the pure strategies of the opponent

Summary Theorems

Here are the three most important theorems of game theory…

- ***Minimax Theorem***
  - Proved by Von Neumann in 1928.
  - States that any 2-player zero-sum matrix game has exactly one unique equilibrium point (which is always a saddle point).
- ***Nash's Theorem***
  - Proved by John Nash in 1951.
  - States that any n-player variable-sum matrix game has at least one equilibrium point.

- ***Theorem (value and saddle point)***

The zero-sum game has a value V if and only if a saddle point exists.